Limit theorems for a general stochastic rumour model

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7 ERPEM
Santa Fe, December 2nd 2010

1Supported by FAPESP (10/06967-2)
The classical stochastic rumour models

The two classical models for the spreading of a rumour are:

1. the Daley-Kendall model (DK model) introduced in 1965;
2. the Maki-Thompson model (MT model) introduced in 1973.

In both models a closed homogeneously mixing population of $N + 1$ individuals is subdivided into three classes:

- ignorants: individuals who are ignorant of the rumour;
- spreaders: individuals who are actively spreading the rumour;
- stiflers: individuals who know the rumour but have ceased spreading it.

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*Both models consider initially 1 spreader and $N$ ignorants.*
In the MT model the rumour is propagated through the population by directed contact between spreaders and other individuals. Then

- when a spreader interacts with an ignorant, the ignorant becomes a spreader;

- whenever a spreader contacts a stifler, the spreader turns into a stifler;

- when a spreader meets another spreader, the initiating one becomes a stifler.
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We denote,

- by $X(t)$ the number of *ignorants* at time $t$,
- by $Y(t)$ the number of *spreaders* at time $t$,
- by $Z(t)$ the number of *stiflers* at time $t$.

Then, the process $\{(X(t), Y(t))\}_{t \geq 0}$ is a CTMC with transitions and corresponding rates given by

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- Watson (1988): generalizes the last result using normal asymptotic approximation (MT and DK model)

- Daley and Gani (1999): analyses the $(\alpha, p)$ version of the DK model

- other variations are introduced by Pearce (2000), Hayes (2005) and Kawachi (2008).

Our contribution

- we introduce the $(\alpha, p, q)$ version for the DK model;

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The \((\alpha, p, q)\)-DK model

Let \(\alpha, p, q \in (0, 1]\) and suppose that, independently,

- a spreader involved in a meeting decides to tell the rumour with probability \(p\);
- once such a decision is made, any spreader in a meeting with somebody informed has probability \(\alpha\) of becoming a stifler;
- upon hearing the rumour, an ignorant becomes a spreader or a neutral with resp. probabilities \(q\) and \(1 - q\).

Then, if \(U(t)\) is the number of neutrals at time \(t\), the CTMC \((X(t), U(t), Y(t))\) evolves according to

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The general model

- Let $V(t) = (X(t), U(t), Y(t))$.
- Initially, $X(0) = N$, $U(0) = 0$, $Y(0) = 1$ and $Z(0) = 0$, and $X(t) + U(t) + Y(t) + Z(t) = N + 1$ for all $t$.
- We suppose that $\{(X(t), U(t), Y(t))\}_{t \geq 0}$ is a CTMC with initial state $(N, 0, 1)$ and

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The general model

We define $\theta = \theta_1 + \theta_2 - \gamma$ and assume that

$$\lambda > 0, \gamma > 0, \theta_1 \geq 0, \theta_2 \geq 0, 0 < \delta \leq 1 \text{ and } 0 \leq \theta \leq 1.$$  \hspace{1cm} (1)

### Remark

Stochastic rumour models reported in the literature:

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<td>DK (1965)</td>
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Some definitions

**Definition**

For $0 < \theta < 1$, consider the function $f_\theta : [0, 1] \rightarrow \mathbb{R}$ given by

$$f_\theta(x) = \frac{(\gamma + \delta \theta)x^\theta - (\gamma + \delta)\theta x - \gamma(1 - \theta)}{\theta(1 - \theta)}.$$ 

For $\theta = 0$ and $\theta = 1$, consider the functions $f_0$ and $f_1$ defined on $(0, 1]$ by

$$f_0(x) = (\gamma + \delta)(1 - x) + \gamma \log x,$$

$$f_1(x) = -\gamma(1 - x) - (\gamma + \delta)x \log x.$$  

For each $\theta$, we define $x_\infty = x_\infty(\delta, \gamma, \theta)$ as the unique root of $f_\theta$ in the interval $(0, 1)$. 

Pablo Martín Rodríguez
Our results: Law of large numbers

Theorem

Assume (1) and let \( x_\infty \) be as in last definition. Define

\[
u_\infty = (1 - \delta)(1 - x_\infty).
\]

Then,

\[
\lim_{N \to \infty} \frac{X^{(N)}(\tau^{(N)})}{N} = x_\infty
\]

and

\[
\lim_{N \to \infty} \frac{U^{(N)}(\tau^{(N)})}{N} = u_\infty
\]

in probability.
Our results: Central limit theorem

**Theorem**

We assume (1). Then,

\[
\sqrt{N} \left( \frac{X^{(N)}(\tau^{(N)})}{N} - x_{\infty}, \frac{U^{(N)}(\tau^{(N)})}{N} - u_{\infty} \right) \overset{D}{\rightarrow} N_2(0, \Sigma)
\]

as \(N \to \infty\), where \(N_2(0, \Sigma)\) is the bivariate normal distribution with mean zero and covariance matrix \(\Sigma\) given by

\[
\Sigma_{11} = x_{\infty}(1 - x_{\infty}) + A^2C,
\]

\[
\Sigma_{12} = -(1 - \delta)\Sigma_{11} + AB,
\]

\[
\Sigma_{22} = (1 - \delta)^2\Sigma_{11} + (1 - \delta)(\delta(1 - x_{\infty}) - 2AB).
\]

**Remark**

\(A, B\) and \(C\) can be computed as functions of our parameters. See Lebensztayn et al. (2010).
Idea for the proofs

The proofs of the results rely on the theory of dependent Markov chains used by Kurtz et al. (2008) in the context of interacting random walks on the complete graph.

In brief, the stochastic process after a suitable acceleration converges to a deterministic system governed by a set of differential equations.

Figura: Behaviour of the MT model after acceleration.
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Figura: Behaviour of the MT model after acceleration.
Let $\rho \in [0, 1]$ and consider our model with the choice $\lambda = \delta = \gamma = 1$, $\theta_1 = \rho$ and $\theta_2 = 1 - \rho$, so $\theta = 0$.

Thus, the limiting proportion of ignorants and the variance of the asymptotic normal distribution in the CLT are given respectively by

\[ x_\infty = x_\infty(1, 1, 0) = - \frac{W_0(-2e^{-2})}{2} \approx 0.203188, \quad \text{and} \]

\[ \sigma^2 = \frac{x_\infty(1 - x_\infty)(1 - 2x_\infty + 2\rho x_\infty^2)}{(1 - 2x_\infty)^2} \approx 0.272736 + 0.0379364 \rho. \]

We obtain MT or DK model accordingly as $\rho$ equals 0 or 1, showing that our theorems generalize the results presented by Sudbury (1985) and Watson (1988).
Let \( \rho \in [0, 1] \) and consider our model with the choice \( \lambda = \delta = \gamma = 1, \) \( \theta_1 = \rho \) and \( \theta_2 = 1 - \rho, \) so \( \theta = 0. \)

Thus, the limiting proportion of ignorants and the variance of the asymptotic normal distribution in the CLT are given respectively by

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Example

The \((\alpha, p, q)\)-DK model is obtained by the choice

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\lambda = p, \quad \delta = q, \quad \theta_1 = \alpha^2(2 - p), \quad \theta_2 = \alpha(1 - \alpha)(2 - p), \quad \text{and} \quad \gamma = \alpha
\]

If \(0 < \alpha(1 - p) < 1\), then \(x_\infty\) is the unique root of the function

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f^*(x) = \frac{(1 + q(1 - p))x^{\alpha(1-p)} - (\alpha + q)(1 - p)x - 1 + \alpha(1 - p)}{(1 - p)(1 - \alpha(1 - p))}
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References


